Lecture 24: Variational Methods Teaser

- In several arguments, we used the idea of energy to obtain a solution or its uniqueness, we use this reasoning again.
- If a system is in equilibrium, it should have 0 kinetic energy and minimum potential energy. We then reformulate PDEs as minimization problems.
- For a bounded domain $U \subseteq \mathbb{R}^n$, $w \in C^2(\bar{U})$, define the Dirichlet Energy

$$\mathcal{E}[w] = \frac{1}{2}\int_U |\nabla w|^2 dx$$

Let us suppose that $u \in C^2(\bar{U}, \mathbb{R})$ satisfies

$$\mathcal{E}[u] \leq \mathcal{E}[u+\varphi] \quad \text{for all } \varphi \in C_c^\infty(U).$$

Then, for $t \in \mathbb{R}$,

$$\frac{d}{dt}\mathcal{E}[u+t\varphi]\Big|_{t=0} = 0$$

where

$$\frac{d}{dt}\mathcal{E}[u+t\varphi]\Big|_{t=0} = \frac{1}{2}\frac{d}{dt}\int_U |\nabla u|^2 + 2t\nabla u\cdot\nabla\varphi + t^2|\nabla\varphi|^2 dx\Big|_{t=0}$$

$$= \int_U \nabla u \cdot \nabla\varphi\, dx$$

$$= -\int_U \varphi\,\Delta u\, dx$$

So this is saying $\Delta u = 0$ in U (Laplace Eqn!)

The Poisson Equation

- Gauss' Law describes ~~the presence of~~ an electrical field in the presence of a charge distribution. It states that the outward flux of the field through a surface is proportional to the total electric field contained by the surface
  - Let U be a C^1 bdd domain, ρ the charge density, E the field

$$\int_{\partial U} E \cdot \eta \, dS = 4\pi k \int_U \rho \, dx \qquad k \in \mathbb{R} \text{ is a constant}$$

then, $4\pi k \int_U \rho \, dx = \int_U \nabla \cdot E \, dx$, which holds for all such U iff $4\pi k \rho = \nabla \cdot E$

Since E is conservative, there is a potential function ϕ so $E = -\nabla \phi$ and the above gives

$$-\Delta \phi = 4\pi k \rho$$

General Poisson Eqn: $\quad -\Delta \phi = f$

Dirichlet's Principle

- We solve $\begin{cases} -\Delta u = f & \text{in } U \\ u|_{\partial U} = 0 \end{cases}$ via minimization for $f \in L^2(U)$ and $u \in \mathcal{H}_0^1(U)$ via the weak formulation

$$\int_U \nabla u \cdot \nabla \psi - f \psi \, dx = 0 \qquad \text{for all } \psi \in C_c^\infty(U).$$

Rmk: Why $\mathcal{H}_0^1$ instead of, say, $\mathcal{H}^2$? Simply to encode $u|_{\partial U} = 0$ while remaining in a Hilbert space.

- Define $D_f[w] = \mathcal{E}[w] - \langle f, w \rangle = \int_U |\nabla w|^2 - f w \, dx$ for $f \in L^2$, $w \in \mathcal{H}_0^1$.

$\boxed{\text{Th}^{\underline{m}}\ 11.1}$ Dirichlet's Principle

Suppose $U \subseteq \mathbb{R}^n$ is a bdd domain and $f \in L^2(U;\mathbb{R})$.
If $u \in H_0^1(U;\mathbb{R})$ satisfies

$$D_f[u] \leq D_f[w]$$

for all $w \in H_0^1(U;\mathbb{R})$, then u is a weak solution of the Poisson equation.

$\boxed{\text{Pf}}$ Since $C_c^\infty \subseteq H_0^1$, $D_f[u] \leq D_f[u+t\varphi]$ for any $\varphi \in C_c^\infty$.

Therefore,

$$0 = \frac{d}{dt} D_f[u+t\varphi]\Big|_{t=0} = \frac{d}{dt}\left(\mathcal{E}[u+t\varphi] - \langle f, u+t\varphi\rangle\right)\Big|_{t=0}$$

$$= \int_U \nabla u \cdot \nabla\varphi - f\varphi\,dx. \qquad \square$$

- In essence, we reduce the PDE to a quadratic minimization, which was based on a similar, more complex argument of Poincare:

$\boxed{\text{Th}^{\underline{m}}}$ Poicare's Inequality.

For a bdd. domain $U \subseteq \mathbb{R}^n$, there is a constant $\kappa > 0$ depending only on U such that

$$\|u\|_2^2 \leq \kappa^2\,\mathcal{E}[u]$$

for all $u \in H_0^1(U)$.

- The constant involved relates directly to the lowest eigenvalue of Δ on U.

- Now, looking at the H^1-norm, Poincare's Inequality says

$$\|u\|_{H^1}^2 = \|u\|_2^2 + \mathcal{E}[u] \leq (\kappa^2+1)\,\mathcal{E}[u]$$

$\rightsquigarrow$ This is to say that $\dfrac{\mathcal{E}[u]}{\|u\|^2} \geq \dfrac{1}{(\kappa^2+1)}$, or that the ratio of the quadratic to the norm is bounded below.

This is called <u>Coercivity.</u>

• Notice that $\mathcal{E}[u] \leq \|u\|_{\mathcal{H}^1}^2$ as well, so $\mathcal{E}[u]$ is a bounded functional in $\mathcal{H}_0^1$. Together,

$$\frac{1}{k^2+1} \leq \frac{\mathcal{E}[u]}{\|u\|^2_{\mathcal{H}_0^1}} \leq 1$$

$\boxed{\text{Th}^{\underline{m}}}$ For a bounded domain $U \subseteq \mathbb{R}^n$ and $f \in L^2(U)$, there is a unique $u \in \mathcal{H}_0^1(U)$ such that

$$D_f[u] \leq D_f[w]$$

for all $w \in \mathcal{H}_0^1(U)$.

$\boxed{\text{Pf}}$ By the Reverse Triangle Inequality,

$$D_f[w] \geq \mathcal{E}[w] - |\langle f, w\rangle| \geq \frac{1}{k^2+1}\|w\|_{\mathcal{H}^1}^2 - \|f\|_2\|w\|_2$$

$$\geq \frac{1}{k^2+1}\|w\|_{\mathcal{H}^1}^2 - \|f\|_2\|w\|_{\mathcal{H}^1}$$

the RHS may be written in the form $cx^2 - bx$ for $x = \|w\|_{\mathcal{H}^1}$.

$$\min_{x \in \mathbb{R}} (cx^2 - bx) = -\frac{b}{4c} \quad \text{for } c > 0, \text{ so}$$

$$D_f[w] \geq -\frac{k^2+1}{4}\|f\|_2^2 \quad \text{for } w \in \mathcal{H}_0^1.$$

If we set $d_0 = \inf_{w \in \mathcal{H}_0^1(U)} D_f[w]$, we know that $d_0 > -\infty$.

Pick some $w_k \in \mathcal{H}_0^1(U)$ so $D_f[w_k] \to d_0$ as $k \to \infty$.

We show $\{w_k\}$ is Cauchy, so that since $\mathcal{H}_0^1(U)$ is complete, there exists some limit and hence a minimizer of D_f.

By direct calculation

$$\mathcal{E}\left[\frac{u+v}{2}\right] = \frac{1}{2}\mathcal{E}[u] + \frac{1}{2}\mathcal{E}[v] - \frac{1}{4}\mathcal{E}[u-v]$$

for all $u, v \in \mathcal{H}_0^1$

• Applying this to D_f,

$$D_f\left[\frac{w_k+w_m}{2}\right] = \frac{1}{2}D_f[w_k] + \frac{1}{2}D_f[w_m] - \frac{1}{4}\mathcal{E}[w_k - w_m] \geq d_0$$

Such that

$$\mathcal{E}[w_k - w_m] \leq 2D_f[w_k] + 2D_f[w_m] - 4d_0$$

and $\lim\limits_{k,m\to\infty} 2D_f[w_k] + 2D_f[w_m] - 4d_0 = 0$

by the choice of $\{w_k\}$. Since $\mathcal{E}[w_k - w_m] \geq 0$,

$$0 \leq \lim_{k,m\to\infty} \|w_k - w_m\|_{2,1} \leq \lim_{k,m\to\infty} \sqrt{(k^2+1)\,\mathcal{E}[w_k - w_m]} = 0$$

• By completeness, let ~~u~~ $u = \lim\limits_{k\to\infty} w_k$, so
$D_f[u] = \lim\limits_{k\to\infty} D_f[w_k] = d_0$ and u minimizes D_f.

• For uniqueness, let u_1, u_2 both have $D_f[u_i] = d_0$.
Since $d_0 = \frac{1}{2}D_f[u_1] + \frac{1}{2}D_f[u_2] - \frac{1}{4}\mathcal{E}[u_1 - u_2]$
as above, $\mathcal{E}[u_1 - u_2] = 0$, so $\|u_1 - u_2\|_2 = 0$. $\square$

⇝ Thus, we obtain a weak solution $u \in H_0^1$ to the Poisson equation.

⇝ This may be applied to a large general class of PDEs under appropriate assumptions.